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DISJOINT FREQUENCY BANDS OF $\nabla_{\omega}^{2n} \neq (\nabla_{\omega}^{2})^{n}$ WAVELETS

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ABSTRACT

We construct a family of orthonormal bases in signal space, where each basis is generated by the sinc function but does not include it. Each function in the n^{th} orthonormal basis is $O(|t|^{-n-1})$ for time t in the distant future or distant past, while the basis is a wavelet basis in the sense that it has dyadic scale coherence. Indeed, wavelets on different length scales are not only orthogonal but have disjoint frequency bands. This property, together with the time-decay property, is obtained through *mixed* translational generation at a given scale. Such a basis has no practical application to signal processing, since complex translations of the sinc function are included in the mixtures – with the directions dictated by the $2n^{\text{th}}$ roots of unity in the complex plane. Our construction is motivated by curiosity, and the calculation of the coefficients for normalized mixtures of translations is a computational challenge that grows with n. We carry out calculation for $n = 2$.

INTRODUCTION

For fixed positive integer n and for $j = 0,1, \dots ,2n - 1$, consider the function β_j with the power series representation

$$\beta_j(\varepsilon) = \sum_{\nu=0}^{\infty}(-1)^{n\nu}\frac{\varepsilon^{2n\nu+j}}{(2n\nu+j)!} \tag{1}$$

In the case $n = 1$, we obviously have $\beta_0(\varepsilon) = \cos\varepsilon$ and $\beta_1(\varepsilon) = \sin\varepsilon$. In the case $n = 2$, we have

$$\beta_0(\varepsilon) = \frac{1}{2}(\cos\varepsilon + \cosh\varepsilon) \tag{2a}$$

$$\beta_1(\varepsilon) = \frac{1}{2}(\sin\varepsilon + \sinh\varepsilon) \tag{2b}$$

$$\beta_2(\varepsilon) = \frac{1}{2}(\cosh\varepsilon - \cos\varepsilon) \tag{2c}$$

$$\beta_3(\varepsilon) = \frac{1}{2}(\sinh\varepsilon - \sin\varepsilon) \tag{2d}$$

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Now, introduce the $n \times n$ matrix-valued function

$$B = \begin{bmatrix} \beta_n & \beta_{n+1} & \beta_{n+2} & \cdots & \beta_{2n-1} \\ \beta_{n-1} & \beta_n & \beta_{n+1} & \cdots & \beta_{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_n \end{bmatrix}, \quad (3)$$

(Note that every function β_j occurs in the matrix except β_0)

Conjecture: If $\varepsilon = \varepsilon_0$ is a positive zero of det $B(\varepsilon)$, then the rank of $B(\varepsilon_0)$ is $n-1$..

This statement is vacuous in the case $n = 1$. In the case $n = 2$,

$$B(\varepsilon) = \frac{1}{2}\begin{bmatrix} \cosh\varepsilon - \cos\varepsilon & \sinh\varepsilon - \sin\varepsilon \\ \sin\varepsilon + \sinh\varepsilon & \cosh\varepsilon - \cos\varepsilon \end{bmatrix} \quad (4)$$

and our conjecture reduces to the observation that $B(\varepsilon)$ is never the zero matrix for $\varepsilon > 0$. Indeed,

$$\sinh\varepsilon > \sin\varepsilon, \quad \varepsilon > 0, \quad (5)$$

and incidentally,

$$\det B(\varepsilon) = \frac{1}{4}[(\cosh\varepsilon - \cos\varepsilon)^2 - (\sinh^2\varepsilon - \sin^2\varepsilon)]$$
$$= \frac{1}{2}[1 - \cos\varepsilon\cosh\varepsilon]. \quad (6)$$

The zeros of det $B(\varepsilon)$ are the solutions of the equation

$$\cos\varepsilon = \operatorname{sech}\varepsilon. \quad (7)$$

Remark 1: For arbitrary $n \in \mathbb{Z}^+$, it is worth pointing out that B is the Wronskian matrix of $(\beta_n, \beta_{n+1}, \dots, \beta_{2n-1})$ but this observation is not enough to resolve the issue.

Our conjecture implies that the solutions of a certain eigenvalue problem are non-degenerate. While this assumption is not essential to our wavelet construction, it simplifies the description of the idea for arbitrary $n \in \mathbb{Z}^+$.

Among all bases of *band-limited* wavelets that have ever been constructed for signal processing, the Meyer basis [1] is arguably the most useful – and certainly the most remarkable. The mother wavelet is not only band-limited but also a Schartz function of time – in addition to generating an *orthonormal* basis of $L^2(-\infty, \infty)$ through dyadic scaling and scale-commensurate translation in time.